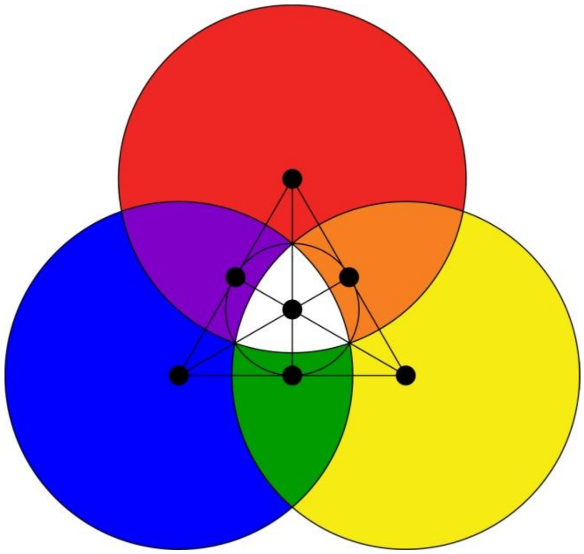

Mathematical Notes

For the interested, mathematically-inclined reader, we outline the algebraic properties of *Al-Jabar*. This section is in no way essential for gameplay. Rather, the following notes are included to aid in analyzing and extending the game rules, which were derived using general formulas, to include sets having any number of "primary" elements, or comprised of game pieces other than colors.

The arithmetic of *Al-Jabar* in the group of the eight colors of the game is isomorphic to the addition of ordered triples in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, that is, 3-vectors whose elements lie in the congruence classes modulo 2.

The relationship becomes clear if we identify the three primary colors red, yellow, and blue with the ordered triples

$$R = (1,0,0),\ Y = (0,1,0),\ B = (0,0,1)$$

and define the black/clear color to be the identity vector

$$K = (0,0,0).$$

We identify the other colors in the game with the following ordered triples using component-wise vector addition:

$$O = R + Y = (1,0,0) + (0,1,0) = (1,1,0)$$

$$G = Y + B = (0,1,0) + (0,0,1) = (0,1,1)$$

$$P = R + B = (1,0,0) + (0,0,1) = (1,0,1)$$

$$W = R + Y + B$$
$$= (1,0,0) + (0,1,0) + (0,0,1) = (1,1,1).$$

The color-addition properties of the game follow immediately from these identities if we sum the vector entries using addition modulo 2. Then the set of colors {R, Y, B, O, G, P, W, K} is a group under the given operation of addition, for it is closed, associative, has an identity element (K), and each element has an inverse (itself).

Certain rules of gameplay were derived from general formulas, the rationale for which involved a mixture of probabilistic and strategic considerations. Using these formulas, the rules of *Al-Jabar* can be generalized to encompass different finite cyclic groups and different numbers of primary elements, i.e. using n-vectors with entries in $\mathbb{Z}_m$, that is, elements of

$$\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_m \times \ldots \times \mathbb{Z}_m \ \ (n \text{ times}).$$

In such a more general setting, there are m "primary" n-vectors of the forms $(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1)$, and the other nonzero m-vectors comprising the group are generated using component-wise addition modulo m, as above. Also, the analog to the black/clear game piece is the zero-vector $(0,0,0,\ldots,0)$.

In addition, the following numbered rules from the Rules of Play would be generalized as described here:

2. The initial pool of game pieces used to deal from will be composed of at least

$$Am^n - A$$

pieces, where A is at least as great as m multiplied by the number of players. This pool of pieces will be divided into an equal number A of every game piece color except for the black/clear or identity-element $(0,0,0,\ldots,0)$. Players will recall that the number of black/clear pieces is arbitrary and intended to be unlimited during gameplay, so this number will not be affected by the choice of m and n;

3. The number of pieces initially dealt to each player will be

$$m^{n+1} - m - 1;$$

5. On each turn, a player will exchange up to n pieces from his or her hand for up to n marbles from the Center with the same sum. The exception to this is the Spectrum, which will consist of the n primary colors

$$(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1)$$

together with the n-vector

$$(m-1, m-1, m-1, \ldots, m-1),$$

which is the generalized analog to the white game piece used in the regular game. It will be seen that these $n+1$ marbles have a sum of $(0,0,0,\ldots,0)$ or black/clear.

A player must draw additional marbles if he or she has more than n pieces in hand and cannot make a move;

6. The cancellation rule will apply to m-tuples (instead of doubles) of identical non-black/clear colors;

8. The first player to have only one piece remaining after his or her turn will signal the final round, or any player having n or fewer pieces in hand may choose to do so.

Thus for the group

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

we have $m = 2, n = 4$ and let $A = 10$. Then each player starts with 29 game pieces dealt from a bag of 10 each of the 15 non-black/clear colors, may exchange up to 4 pieces on any turn or 5 pieces in the case of a Spectrum move, and may signal the end of the game with 4 or fewer pieces in hand.

Here the Spectrum consists of the colors

$$(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,1,1)$$

and the cancellation rule still applies to doubles in this example, as $m = 2$.

Other cyclic groups may also be seen as sets of colors under our addition, such as

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$$

in which every game piece either contains, for example, no red (0), light red (1) or dark red (2) in the first vector entry, and either contains no blue (0), light blue (1) or dark blue (2) in the second entry. Therefore we might respectively classify the nine elements above as the set

{black/clear, light blue, dark blue,
light red, light purple, bluish purple,
dark red, reddish purple, dark purple}.

Of course, other colors rather than shades of red and blue may be used, or even appropriately selected non-colored game pieces.

Further generalizations of the game rules may be possible—for instance, using n-vectors in $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \ldots \times \mathbb{Z}_{a_n}$ where the subscripts a_i are not all equal—and new games might be produced by other modifications to the rules of play or the game pieces used.

Note that the *Al-Jabar* logo encodes the algebra of the game. Each node on the Fano plane diagram represents the color on which it falls; the sum of any two nodes lying on the same line segment (or on the inner circle) is equal to the third node on that segment.